A Course in Mathematical Physics 1 and 2

Walter Thirring

A Course in Mathematical Physics

1 and 2

Classical Dynamical Systems and Classical Field Theory

Second Edition

Translated by Evans M. Harrell

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Summary of Contents

Walter Thirring

A Course in Mathematical Physics

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Classical Dynamical Systems

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Preface to the Second Edition

The last decade has seen a considerable renaissance in the realm of classical dynamical systems, and many things that may have appeared mathematically overly sophisticated at the time of the first appearance of this textbook have since become the everyday tools of working physicists. This new edition is intended to take this development into account. I have also tried to make the book more readable and to eradicate errors.

Since the first edition already contained plenty of material for a one-semester course, new material was added only when some of the original could be dropped or simplified. Even so, it was necessary to expand the chapter with the proof of the K–A–M Theorem to make allowances for the current trend in physics. This involved not only the use of more refined mathematical tools, but also a reevaluation of the word "fundamental." What was earlier dismissed as a grubby calculation is now seen as the consequence of a deep principle. Even Kepler's laws, which determine the radii of the planetary orbits, and which used to be passed over in silence as mystical nonsense, seem to point the way to a truth unattainable by superficial observation: The ratios of the radii of Platonic solids to the radii of inscribed Platonic solids are irrational, but satisfy algebraic equations of lower order. These irrational numbers are precisely the ones that are the least well approximated by rationals, and orbits with radii having these ratios are the most robust against each other's perturbations, since they are the least affected by resonance effects. Some surprising results about chaotic dynamics have been discovered recently, but unfortunately their proofs did not fit within the scope of this book and had to be left out.

In this new edition I have benefited from many valuable suggestions of colleagues who have used the book in their courses. In particular, I am deeply grateful to H. Grosse, H.-R. Grümm, H. Narnhofer, H. Urbantke, and above

all M. Breitenecker. Once again the quality of the production has benefited from drawings by R. Bertlmann and J. Ecker and the outstanding word-processing of F. Wagner. Unfortunately, the references to the literature have remained sporadic, since any reasonably complete list of citations would have overwhelmed the space allotted.

Vienna, July, 1988 Walter Thirring

Preface to the First Edition

This textbook presents mathematical physics in its chronological order. It originated in a four-semester course I offered to both mathematicians and physicists, who were only required to have taken the conventional introductory courses. In order to be able to cover a suitable amount of advanced material for graduate students, it was necessary to make a careful selection of topics. I decided to cover only those subjects in which one can work from the basic laws to derive physically relevant results with full mathematical rigor. Models which are not based on realistic physical laws can at most serve as illustrations of mathematical theorems, and theories whose predictions are only related to the basic principles through some uncontrollable approximation have been omitted. The complete course comprises the following one-semester lecture series:

I. Classical Dynamical Systems
II. Classical Field Theory
III. Quantum Mechanics of Atoms and Molecules
IV. Quantum Mechanics of Large Systems

Unfortunately, some important branches of physics, such as the relativistic quantum theory, have not yet matured from the stage of rules for calculations to mathematically well understood disciplines, and are therefore not taken up. The above selection does not imply any value judgment, but only attempts to be logically and didactically consistent.

General mathematical knowledge is assumed, at the level of a beginning graduate student or advanced undergraduate majoring in physics or mathematics. Some terminology of the relevant mathematical background is collected in the glossary at the beginning. More specialized tools are introduced as they are needed; I have used examples and counterexamples to try to give the motivation for each concept and to show just how far each assertion may be applied. The best and latest mathematical methods to appear on the market have been used whenever possible. In doing this many an old and trusted favorite of the older generation has been forsaken, as I deemed it best not to hand dull and worn-out tools down to the next generation. It might perhaps seem extravagant to use manifolds in a treatment of Newtonian mechanics, but since the language of manifolds becomes unavoidable in general relativity, I felt that a course that used them right from the beginning was more unified.

References are cited in the text in square brackets [ ] and collected at the end of the book. A selection of the more recent literature is also to be found there, although it was not possible to compile a complete bibliography.

I am very grateful to M. Breitenecker, J. Dieudonné, H. Grosse, P. Hertel, J. Moser, H. Narnhofer, and H. Urbantke for valuable suggestions. F. Wagner and R. Bertlmann have made the production of this book very much easier by their greatly appreciated aid with the typing, production and artwork.

Walter Thirring

Note about the Translation

In the English translation we have made several additions and corrections to try to eliminate obscurities and misleading statements in the German text. The growing popularity of the mathematical language used here has caused us to update the bibliography. We are indebted to A. Pflug and G. Siegl for a list of misprints in the original edition. The translator is grateful to the Navajo Nation and to the Institute for Theoretical Physics of the University of Vienna for hospitality while he worked on this book.

Evans M. Harrell

Walter Thirring

Contents of Volume 1

Glossary

Logical Symbols

$\forall$	for every
$\exists$	there exist(s)
$\nexists$	there does not exist
$\exists!$	there exists a unique
$a \Rightarrow b$	if a then b
iff	if and only if

Sets

$a \in A$	a is an element of A
$a \notin A$	a is not an element of A
$A \cup B$	union of A and B
$A \cap B$	intersection A and B
CA	complement of A (In a larger set B: $\{a: a \in B, a \notin A\}$)
$A \backslash B$	$\{a: a \in A, a \notin B\}$
$A \Delta B$	symmetric difference of A and B: $(A \backslash B) \cup (B \backslash A)$
$\varnothing$	empty set
$C\varnothing$	universal set
$A \times B$	Cartesian product of A and B: the set of all pairs (a, b), $a \in A$, $b \in B$

Important Families of Sets

open sets	contains $\varnothing$ and the universal set and some other specified sets, such that the open sets are closed under union and finite intersection

closed sets	the complements of open sets
measurable sets	contains $\varnothing$ and some other specified sets, and closed under complementation and countable intersection
Borel-measurable sets	the smallest family of measurable sets which contains the open sets
null sets, or sets of measure zero	the sets whose measure is zero. "Almost everywhere" means "except on a set of measure zero."

An equivalence relation is a covering of a set with a non-intersecting family of subsets. $a \sim b$ means that a and b are in the same subset. An equivalence relation has the properties: i) $a \sim a$ for all a. ii) $a \sim b \Rightarrow b \sim a$. iii) $a \sim b, b \sim c \Rightarrow a \sim c$.

Numbers

$\mathbb{N}$	natural numbers
$\mathbb{Z}$	integers
$\mathbb{R}$	real numbers
$\mathbb{R}^+(\mathbb{R}^-)$	positive (negative) numbers
$\mathbb{C}$	complex numbers
sup	supremum, or lowest upper bound
inf	infimum, or greatest lower bound
I	any open interval
(a, b)	the open interval from a to b
$[a, b]$	the closed interval from a to b
$(a, b]$ and $[a, b)$	half-open intervals from a to b
$\mathbb{R}^n$	$\underbrace{\mathbb{R} \times \cdots \times \mathbb{R}}_{N \text{ times}}$ This is a vector space with the scalar product $(y_1, \ldots, y_N \vert x_1, \ldots, x_N) = \sum_{i=1} y_i x_i$

Maps (=Mappings, Functions)

$f: A \to B$	for every $a \in A$ an element $f(a) \in B$ is specified
$f(A)$	image of A, i.e., if $f: A \to B$, $\{f(a) \in B : a \in A\}$
$f^{-1}(b)$	inverse image of b, i.e. $\{a \in A : f(a) = b\}$
f^{-1}	inverse mapping to f. Warning: 1) it is not necessarily a function 2) distinguish from $1/f$ when $B = \mathbb{R}$
$f^{-1}(B)$	inverse image of B: $\bigcup_{b \in B} f^{-1}(b)$
f is injective (one-to-one)	$a_1 \neq a_2 \Rightarrow f(a_1) \neq f(a_2)$
f is surjective (onto)	$f(A) = B$
f is bijective	f is injective and surjective. Only in this case is f^{-1} a true function
$f_1 \times f_2$	the function defined from $A_1 \times A_2$ to $B_1 \times B_2$, so that $(a_1, a_2) \to (f_1(a_1), f_2(a_2))$
$f_2 \circ f_1$	f_1 composed with f_2: if $f_1: A \to B$ and $f_2: B \to C$, then $f_2 \circ f_1: A \to C$ so that $a \to f_2(f_1(a))$
$\mathbf{1}$	identity map, when $A = B$; i.e., $a \to a$. Warning: do not confuse with $a \to 1$ when $A = B = \mathbb{R}$.

$f\|_u$	f restricted to a subset $U \subset A$
$f\|_a$	evaluation of the map f at the point a; i.e., $f(a)$.
f is continuous	the inverse image of any open set is open
f is measurable	the inverse image of any measurable set is measurable
supp f	support of f: the smallest closed set on whose complement $f = 0$
C^r	the set of r times continuously differentiable functions
C_0^r	the set of C^r functions of compact (see below) support
χ_A	characteristic function of $A: \chi_A(a) = 1\ldots$

Topological Concepts

Topology	any family of open sets, as defined above
compact set	a set for which any covering with open sets has a finite subcovering
connected set	a set for which there are no proper subsets which are both open and closed
discrete topology	the topology for which every set is an open set
trivial topology	the topology for which the only open sets are $\varnothing$ and $C\varnothing$
simply connected set	a set in which every path can be continuously deformed to a point
(open) neighborhood of $a \in A$	any open subset of A containing a. Usually denoted by U or V.
(open) neighborhood of B $\subset A$	any open subset of A containing B
p is a point of accumulation (= cluster point)	for any neighborhood U containing p, $U \cap B \neq \{p\}$ or $\varnothing$
$\bar{B}$	closure of B: the smallest closed set containing B
B is dense in A	$\bar{B} = A$
B is nowhere dense in A	$A \backslash \bar{B}$ is dense in A
metric (distance function) for A	a map $d: A \times A \to \mathbb{R}$ such that $d(a, a) = 0$; $d(a, b) = d(b, a) > 0$ for $b \neq a$; and $d(a, c) \leq d(a, b) + d(b, c)$ for all a, b, c in A. A metric induces a topology on A, in which all sets of the form $\{b : d(b, a) < \eta\}$ are open.
separable space	a space with a countable dense subset
homeomorphism	a continuous bijection with a continuous inverse
product topology on $A_1 \times A_2$	the family of open sets of the form $U_1 \times U_2$, where U_1 is open in A_1 and U_2 is open in A_2, and unions of such sets

Mathematical Conventions

$f_{,i}$	$\partial f / \partial q_i$
$\dot{q}(t)$	$\dfrac{dq(t)}{dt}$
$\det\|M_{ij}\|$	determinant of the matrix M_{ij}
Tr M	$\sum_i M_{ii}$
δ^i_j, δ_{ij}	1 if $i = j$, otherwise 0

$\varepsilon_{i_1,\ldots,i_m}$	the totally antisymmetric tensor of degree m, with values ± 1.
M^t	transposed matrix: $(M^t)_{ij} = M_{ji}$
M^*	Hermitian conjugate matrix: $(M^*)_{ij} = (M_{ji})^*$
$\mathbf{v}\cdot\mathbf{w}$, $(\mathbf{v}\mid\mathbf{w})$, or $(\mathbf{v}\cdot\mathbf{w})$	scalar (inner, dot) product
$\mathbf{v}\times\mathbf{w}$ or $[\mathbf{v}\wedge w]$	cross product
∇f	gradient of f
$\nabla\times\mathbf{f}$	curl of $\mathbf{f}$
$\nabla\cdot\mathbf{f}$	divergence of $\mathbf{f}$
$\Vert\mathbf{v}\Vert$ (in 3 dimensions, $\lvert\mathbf{v}\rvert$)	length of the vector $\mathbf{v}$: $\Vert\mathbf{v}\Vert = (\sum_{i=1} v_i^2)^{1/2} = d(\mathbf{0},\mathbf{v})$
$d\mathbf{s}$	differential line element
$d\mathbf{S}$	differential surface element
$d^m q$	m-dimensional volume element
$\perp$	is perpendicular (orthogonal) to
$\Vert$	is parallel to
$\measuredangle$	angle
$d\Omega$	element of solid angle
$\mathrm{Mat}_n(\mathbb{R})$	the set of real $n\times n$ matrices
$O(x)$	order of x

The summation convention for repeated indices is understood except where it does not make sense. For example, $L_{ik}x_k$ stands for $\sum_k L_{ik}x_k$.

Groups

GL_n	group of $n\times n$ matrices with nonzero determinant
O_n	group of $n\times n$ matrices M with $MM^t = \mathbf{1}$ (unit matrix)
SO_n	subgroup of O_n with determinant 1
E_n	Euclidean group
S_n	group of permutations of n elements
U_n	group of complex $n\times n$ matrices M with $MM^* = \mathbf{1}$ (unit matrix)
Sp_n	group of symplectic $n\times n$ matrices

Physical Symbols

m_i	mass of the i-th particle
$\mathbf{x}_i$	Cartesian coordinates of the i-th particle
$t = x^\circ/c$	time
s	proper time
q_i	generalized coordinates
p_i	generalized momenta
e_i	charge of the i-th particle
κ	gravitational constant
c	speed of light
$\hbar = h/2\pi$	Planck's constant divided by 2π
F^α_β	electromagnetic field tensor
$g_{\alpha\beta}$	gravitational metric tensor (relativistic gravitational potential)
$\mathbf{E}$	electric field strength
$\mathbf{B}$	magnetic field strength in a vacuum
$\sim$	is on the order of
$\gg$	is much greater than

Symbols Defined in the Text

Df	derivative of $f: \mathbb{R}^n \to \mathbb{R}^m$	(2.1.1)
(V, Φ)	chart	(2.1.3)
T^n	n-dimensional torus	(2.1.7; 2)
S^n	n-dimensional sphere	(2.1.7; 2)
∂M	boundary of M	(2.1.20)
$\Theta_C(q)$	mapping of the tangent space into $\mathbb{R}^m$	(2.2.1)
$T_q(M)$	tangent space at the point q	(2.2.4)
$T_q(f)$	derivative of f at the point q	(2.2.7)
$T(M)$	tangent bundle	(2.2.12)
Π	projection onto a basis	(2.2.15)
$T(f)$	derivative of $f: M_1 \to M_2$	(2.2.17)
$\mathscr{T}^1_0(M)$	set of vector fields	(2.2.19)
Φ_*	induced mapping on $\mathscr{T}^r_s$	(2.2.21)
L_X	Lie derivative	(2.2.25; 1), (2.5.7)
∂_i	natural basis on the tangent space	(2.2.26)
Φ^x_t	flow	(2.3.7)
τ^x_t	automorphism of a flow	(2.3.8)
W	action	(2.3.16)
L	Lagrangian	(2.3.17)
H	Hamiltonian	(2.3.26)
$T^*_q(M)$	cotangent space	(2.4.1)
e^*_i	dual basis	(2.4.2; 1)
df	differential of a function	(2.4.3; 1)
$T_{qs}^r(M)$	space of tensors	(2.4.4)
$\otimes$	tensor product	(2.4.5)
$T^r_s(M)$	tensor bundle	(2.4.8)
$\mathscr{T}^r_s(M)$	set of tensor fields	(2.4.28)
df	differential of a function	(2.4.13; 1)
g	pseudo-Riemannian metric	(2.4.27)
$\overset{\pi}{\times}$	fiber product	(2.4.34)

$T^*(\Phi)$	transposed derivative	(2.4.34)
$(\Phi^{-1})^*$	pull-back, or inverse image of the covariant tensors	(2.4.41)
$E_p(M)$	set of p-forms	(2.4.27)
$\wedge$	wedge (outer, exterior) product	(2.4.28)
$*$	$*$-mapping	(2.4.18)
i_X	interior product	(2.4.33)
d	exterior derivative	(2.5.1)
$[\,]$	Lie bracket	(2.5.9; 6)
Θ, ω	canonical forms	(3.1.1)
Ω	Liouville measure	(3.1.2; 3)
X_H	Hamiltonian vector field	(3.1.9)
b	bijection associated with ω	(3.1.9)
$\{\,\}$	Poisson brackets	(3.1.11)
M_e	generalized configuration space	(3.2.12)
$\mathscr{H}$	Hamiltonian on M_e	(3.2.12)
(I, φ)	action-angle variables	(3.3.14)
$\Omega_\pm$	Møller transformations	(3.4.4)
S	scattering matrix	(3.4.9)
$d\sigma$	differential scattering cross-section	(3.4.12)
$\mathbf{L}$	angular momentum	(4.1.3)
$\mathbf{K}$	boost	(4.1.9)
$\eta_{\alpha\beta}$	Minkowski space metric	(5.1.2)
γ	$1/\sqrt{1 - v^2/c^2}$ (relativistic dilatation)	(5.1.2)
F	electromagnetic 2-form	(5.1.10; 1)
A	1-form of the potential	(5.1.10; 1)
Λ	Lorentz transformation	(5.1.12)
r_0	Schwarzschild radius	(5.7.1)

Walter Thirring

A Course in Mathematical Physics

2

Classical Field Theory

Second Edition

Translated by Evans M. Harrell

Preface

In the past decade the language and methods of modern differential geometry have been increasingly used in theoretical physics. What seemed extravagant when this book first appeared 12 years ago, as lecture notes, is now a commonplace. This fact has strengthened my belief that today students of theoretical physics have to learn that language—and the sooner the better. After all, they will be the professors of the twenty-first century and it would be absurd if they were to teach then the mathematics of the nineteenth century. Thus for this new edition I did not change the mathematical language. Apart from correcting some mistakes I have only added a section on gauge theories. In the last decade it has become evident that these theories describe fundamental interactions, and on the classical level their structure is sufficiently clear to qualify them for the minimum amount of knowledge required by a theoretician. It is with much regret that I had to refrain from incorporating the interesting developments in Kaluza-Klein theories and in cosmology, but I felt bound to my promise not to burden the students with theoretical speculations for which there is no experimental evidence.

I am indebted to many people for suggestions concerning this volume. In particular, P. Aichelburg, H. Rumpf and H. Urbantke have contributed generously to corrections and improvements. Finally, I would like to thank Dr. I. Dahl-Jensen for redoing some of the figures on the computer.

Vienna
December, 1985

W. Thirring

Contents of Volume 2